Determinism, Chaos and Quantum Mechanics.

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Abstract

After some general remarks on the notion of "determinism", I will discuss the precise meaning of chaos theory and the frequent misunderstandings concerning the implications of that theory. After reviewing the status of probabilistic reasoning in classical physics, I will also briefly discuss misunderstandings occurring in the context of quantum mechanics.

1. Some General Remarks on Determinism.

Determinism is one of those words over which many people get involved in disputes, partly because it is rarely defined. Hence, it is often unclear what the discussion is *about*. My goal in this section will be to propose two definitions. According to one definition, determinism is trivially false. According to the other, it is very probably true. However, both definitions are rather uninteresting, and I must say that I don't know how to formulate the issue of determinism so that the question becomes interesting. However, there exists a certain hostility , maybe caused by a fear, with respect to the idea of determinism in the educated public, and this hostility tends to have culturally negative effects; it is therefore important to clarify this issue and to dispel the reasons causing this hostility.

The first definition simply equates determinism and predictability. So, according to that definition, a process is deterministic if we, humans, can predict it, or, maybe, if we, humans, will be able to predict it in the future[1]. One problem with that definition is that nobody who has ever defended universal determinism (in particular Laplace, as we will see below) ever meant it to be true in that sense of the word. Everybody agrees that not everything in the world is predictable, and it is somewhat surprising to see how many people present that truism as if it was a recent discovery.

But the main problem with that definition is best illustrated by an example: suppose that we consider a perfectly regular, deterministic *and* in principle predictable mechanism, like a clock, but put it on the top of a mountain, or in a locked drawer, so that its state (its initial conditions) become inaccessible to us. This renders the system trivially unpredictable, yet it seems difficult to claim that it becomes non-deterministic.

So, one has to admit that *some* physical phenomena obey deterministic laws and, yet, are not predictable, possibly for "accidental" reasons. But, once this is admitted, how does one show that *any* unpredictable system is *truly* non-deterministic, and that the lack of predictability is not merely due to some limitation of our knowledge or of our abilities? We cannot infer indeterminism from ignorance alone. One needs other arguments.

In fact, confusing determinism and predictability is an instance of what E. T. Jaynes calls the "Mind Projection Fallacy": "We are all under an ego-driven temptation to project our private thoughts out onto the real world, by supposing that the creations of one's own imagination are real properties of Nature, or that one's own ignorance signifies some kind of indecision on the part of Nature"

[1]In an often quoted lecture to the Royal Society, on the three hundredth anniversary of Newton's Principia, Sir James Lighthill gave a perfect example of how to identify predictability to determinism: "We are all deeply conscious today that the enthusiasm of our forebears for the marvellous achievements of Newtonian mechanics led them to make generalizations in this area of *predictability* which, indeed, we may have generally tended to believe before 1960, but which we now recognize were false. We collectively wish to apologize for having misled the general educated public by spreading ideas about *determinism* of systems satisfying Newton's laws of motion that, after 1960, were to be proved incorrect..." [24] (Italics are mine; quoted e.g. by Reichl [32], p.3, and by Prigogine and Stengers, [31], p.93, and [30], p.41).

[2]([21], p.7).

This brings us to the second definition of determinism, that tries to be independent of human abilities; consider a physical system whose state is characterized by some numbers that change over time; let us say that it is deterministic if there exists a function F that maps the values taken by that set of variables at a given instant, say t_1, to those obtained at a later time, say t_2; and, then, the latter to those at a later time, t_3, etc. This corresponds pretty much to Laplace's conception[3]. It corresponds to the idea of predictability, but "in principle", i.e. putting aside limitations imposed by human abilities. The word 'exist' should be taken here in a "Platonic" sense : it does not refer to our *knowledge*; the function in question may be unknown, or unknowable, or so complicated that, even if it were known, it would be, in practice, impossible to use it in order to make predictions.

Now, is determinism understood in this sense also refutable? Well, let us suppose that the system happens to be twice in exactly the same state, at different times, say at time t_i and t_j, and is, at times t_{i+1} and t_{j+1}, in different states. Then, the function F does not exist, since, by definition, it is supposed to associate to each set of values of the variables another set in a unique way. To see how this problem may occur, consider a simple example, say the production of grain of a given country, counted yearly and in millions of tons. That is a single number and it is quite possible that is takes the same value, say in 1984 and 1993, but different values in 1985 and 1994. In that case, one should say that this system is not deterministic, according the above definition. But suppose we were to count the amount of grain, not in millions of tons, but in grams. Or suppose we consider a larger set of data- say the quantities of all items produced in a given country. Then, it becomes unlikely that *this set of data* takes exactly the same value twice. And, if that was the case, we could always include more data,

[2]Jaynes' criticisms were mostly directed at the way quantum theory is presented, but they also apply to some discussions of chaos theory or of statistical mechanics.

[3]Except that he was speaking in terms of "continuous time", rather than "discrete time", which chosen here because it looks more intuitive.

say the amount of rainfall, the results of elections etc. in order to avoid that situation. But then, how can the mere existence of the function F be refuted? In fact, its existence is trivial- take any finite sequence of sets of numbers that never repeats itself. One can always find one- in fact many- functions that maps each set into the next one.

So, let me propose the following definition: a physical system is *deterministic* if there exists a function F, as described here, whenever we describe it in sufficient detail. But what is the point of this exercise? Simply to show that, if the idea of determinism is phrased in terms similar to those of Laplace, and not confused with the notion of predictability, then, it is indeed hard to see how it could be refuted. As far as we know, there exists only one world and it never occurs twice in exactly the same state (if it is described in sufficient detail).

Of course, the notion of determinism introduced here has very little to do with the goals of science, which is not simply trying to find a function like F. In a sense, scientists do look for such functions, but with extra properties: simplicity, explanatory power, and, of course, the possibility, using F, to make at least some predictions. So, in a sense, the question of the existence of F is "metaphysical" and of no scientific interest. But, so is the question of "determinism": as far as I can see, there is no notion of determinism that would make the question scientifically relevant. Either it is identified with predicatbility, and determinism is trivially false, or it is defined as above, and it is most likely true, but uninteresting. Yet, the word keeps on coming back, most often in contexts where one explains that "determinism" has finally been refuted, is untenable etc.; and to some extent, that cultural fixation is interesting: why does the concept of determinism provoke such hostility and what effect does that hostility have?

At this point, a remark on the use of the word determinism in the social sciences might be in order: it is rather common to hear (at least nowadays) that human behaviour or society or history is "not determined" or "does not obey deterministic laws" etc. Now, what is obvious, and should have been obvious all along, is that human behaviour, society etc. are not predictable. But, before claiming that they are not deterministic in some other sense, one should remark that our description of such 'objects' is always very incomplete. We cannot include in our data the state of every brain cell of every human being on earth, or even a far less detailed description of the world. So that, even if we have a well tested theory of some social phenomenon, the fact that that theory would not lead to deterministic conclusions implies nothing concerning the possible existence of a (more detailed) deterministic description. Actually, that is what happens in macroscopic physics: random processes are used all the time, but, in general, in cases where there exists a more detailed (microscopic) description, and the "randomness" reflects simply the fact that we operate with a reduced number of (macroscopic) variables (like temperature, pressure, density, etc.). I shall return to the status of probabilities in classical physics in Section 3.

It is likely that the hostility to determinism comes from a desire to "save free will". Namely, to find a description of the physical universe that can be reconciled with our deep feeling that, at least on some occasions, "we" choose to do X and not Y. That is, that Y was possible, but did not happen because of our choice. Indeed, the anti-determinists will probably say that, if everything is caused by anterior events, ultimately going back to the Big Bang, Y was not really possible (it only appeared to be so because of our ignorance) and free will is an illusion. Since most of our moral, legal and political philosophies assume some kind of free will, a lot appears to be at stake[4]. But the problem is, what is the alternative to determinism *within physics*? As far as I can see, nothing has ever been proposed except *pure randomness*! Or, in other words, events with no cause. But that will not give us a picture of the world in which free will exists either. Our feeling of free will is not that there is some intrinsically random process at work in our minds, but that *conscious choices* are made. And that is simply something that no known physical theory accounts for. Our feeling of free will implies that there is that is a causal agent in the world, the 'I', that is

[4]Moreover, the philosophies that deny, or are accused to deny, free will, like Marxism, are associated, in the dominant political discourse, with "totalitarianism"; this, of course, adds a strong emotional charge to the whole dicussion.

simply 'above' all physical laws. It suggests a dualistic view of the world, which itself meets great difficulties. One solution is, as mentioned above, to declare that free will is an illusion. But if that is the case, it is a 'necessary illusion' in the sense that we cannot live without, in some sense, believing in it, unlike, say, believing in the dogma of the Immaculate Conception. I do not claim to have any solution to that problem[5]. I simply would like to avoid the problem to backfire into physics and to create there a prejudice in favour of indeterminism.

Indeed, such a prejudice is, largely for confused moral and political reasons, rather widespread. Yet, as Bertrand Russell observed, scientists should look for deterministic laws like mushroom seekers should look for mushrooms. Deterministic laws are preferable to non deterministic ones because they give both a way to control things more efficiently (at least in principle) and because they give more satisfactory explanations of why things are the way they are. Looking for deterministic laws behind the apparent disorder of things is at the heart of the scientific enterprize. Whether we succeed or not depends in a complicated way both on the structure of the world and on the structure of our minds. But the opposition to determinism tends to make people feel that the project itself is doomed to fail; and that state of mind does run counter to the scientific spirit.

I shall now turn to arguments that are supposed to show that determinism is untenable because of recent physical discoveries.

2. Chaos theory and its implications.

2.1. What is chaos theory about?

A rather popular, but misleading, way to answer that question goes as follows: There are many physical phenomena governed by deterministic laws, and therefore predictable in principle, which are nevertheless unpredictable in practice because of their "sensitivity

[5]The philosopher C. McGinn has developped the interesting idea that this problem lies beyond the limits of human understanding [25].

to initial conditions"[6]. In other words, an arbitrarily small error on the initial conditions makes itself felt after a long enough time. This means that two systems obeying the same laws may, at some moment in time, be in very similar (but not identical) states and yet, after a brief lapse of time, find themselves in very different states. This phenomenon is expressed figuratively by saying that a butterfly flapping its wings today in Madagascar could provoke a hurricane three weeks from now in Florida. Of course, the butterfly by itself doesn't do much. But if one compares the two systems constituted by the Earth's atmosphere with and without the flap of the butterfly's wings, the result three weeks from now may be very different (a hurricane or not). One practical consequence of this is that we do not expect to be able to predict the weather more than a few weeks ahead. Indeed, one would have to take into account such a vast quantity of data, and with such a precision, that even the largest conceivable computers could not begin to cope.

But that popular presentation is not precise enough; indeed, almost all processes have the property that "an arbitrarily small error on the initial conditions makes itself felt after a long enough time". Take a pendulum, or even a more modern clock; eventually, it will indicate the wrong time. In fact, for any system, whose initial

[6]Here is a simple example. Consider the "phase space" to be simply the interval $I = [0, 1[$. And take as (discrete time) dynamics the map $f : x \to 10x$ *mod* 1. This means, we take a number between 0 and 1, multiply it by 10, write the result as an integer plus a number between 0 and 1 and take the latter as the result (i.e. $f(x)$). This gives again a number between 0 and 1, and we can repeat the operation. Upon iteration, we obtain the *orbit* of x; x itself is the initial condition. To describe concretely the latter, one uses the decimal expansion. Any number in I can be written as $x = 0.a_1a_2a_3\ldots$, where a_i equals $0, 1, 2, \ldots, 9$. It is easy to see that $f(x) = 0.a_2a_3\ldots$. This is a perfect example of a *deterministic* but *unpredictable* system. Given the state x at some initial time, one has a rule giving the state of the system for arbitrary times. Moreover, for any fixed time, one can, in principle, find the state after that time, with any desired accuracy, given a sufficiently precise characterization of the initial state. This expresses the deterministic aspect. Unpredictability comes from the fact that, if we take two initial conditions at a distance less than 10^{-n}, then the corresponding orbits could differ by, say, 1/2, after n steps, because the difference will be determined by the nth decimal. One of the relatively recent discoveries in dynamical systems is that simple physical examples, like a forced pendulum, may behave more or less like this map.

state is imperfectly known (which is always the case in practice), an imprecision in the initial data will be reflected in the quality of the predictions we are able to make about the system's future state. In general, the predictions will become more inexact as time goes on. But the *manner* in which the imprecision increases differs from one system to another: in some systems it will increase slowly, in others very quickly.[7]

To explain this, let us imagine that we want to reach a certain specified precision in our final predictions, and let us ask ourselves how long our predictions will remain sufficiently accurate. Let us suppose, moreover, that a technical improvement has allowed us to reduce by half the imprecision of our knowledge of the initial state. For the first type of system (where the imprecision increases slowly), the technical improvement will permit us to *double* the length of time during which we can predict the state of the system with the desired precision. But for the second type of system (where the imprecision increases quickly), it will allow us to increase our "window of predictability" by only a fixed amount: for example, by one additional hour or one additional week (how much depends on the circumstances). Simplifying somewhat, we shall call systems of the first kind *non-chaotic* and systems of the second kind *chaotic*. Chaotic systems are therefore characterized by the fact that their predictability is sharply limited, because even a spectacular improvement in the precision of the initial data (for example, by a factor of 1000) leads only to a rather mediocre increase in the duration over which the predictions remain valid.[8]

It is perhaps not surprising that a very complex system, such as the Earth's atmosphere, is difficult to predict. What is more surprising is that a system describable by a *small* number of variables

[7]In technical terms: in the first case the imprecision increases linearly or polynomially with time, and in the second case exponentially. An example of the latter is given in the previous footnote.

[8]It is important to add one qualification: for some chaotic systems, the fixed amount that one gains when doubling the precision in the initial measurements can be very long, which means that in practice these systems can be predictable much longer than most non-chaotic systems. For example, recent research has shown that the orbits of some planets have a chaotic behavior, but the "fixed amount" is here of the order of several million years.

and obeying simple deterministic equations — for example, a pair of pendulums attached together — may nevertheless exhibit very complex behavior and an extreme sensitivity to initial conditions.

However, one should avoid jumping to hasty philosophical conclusions. For example, it is frequently asserted that chaos theory has shown the limits of science. But many systems in Nature are non-chaotic; and even when studying chaotic systems, scientists do not find themselves at a dead end, or at a barrier which says "forbidden to go further". Chaos theory opens up a vast area for future research and draws attention to many new objects of study.[9] Besides, thoughtful scientists have always known that they cannot hope to predict or compute *everything*. It is perhaps unpleasant to learn that a specific object of interest (such as the weather three weeks hence) escapes our ability to predict it, but this does not halt the development of science. For example, physicists in the nineteenth century knew perfectly well that it is impossible in practice to know the positions of all the molecules of a gas. This spurred them to develop the methods of statistical physics, which have led to an understanding of many properties of systems (such as gases) that are composed of a large number of molecules (see Section 4). Similar statistical methods are employed nowadays to study chaotic phenomena. And, most importantly, the aim of science is not only to predict, but also to understand.

A second confusion concerns Laplace and determinism. Laplace's work is often misunderstood. When he introduced the concept of universal determinism[10], he immediately added that *we* shall "always remain infinitely removed" from this imaginary "intelligence" and its ideal knowledge of the "respective situation of the beings who compose" the natural world, that is, in modern language, of the precise initial conditions of all the particles. He distinguished clearly between what Nature does and the knowledge we have of

[9] Strange attractors, Lyapunov exponents, etc.

[10] "Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes." (Laplace, [22], p. 4)

it. Moreover, he stated this principle at the beginning of an essay on *probability theory*. But, what is probability theory for Laplace? Nothing but a method that allows us to reason in situations of partial ignorance (see Section 3 for a more detailed discussion). The meaning of Laplace's text is completely misrepresented if one imagines that *he* hoped to arrive someday at a perfect knowledge and a universal predictability, for the aim of his essay was precisely to explain how to proceed in the absence of such a perfect knowledge — as one does, for example, in statistical physics.

Over the past three decades, remarkable progress has been made in the mathematical theory of chaos, but the idea that some physical systems may exhibit a sensitivity to initial conditions is not new. And, with regard to meteorological predictions, Henri Poincaré in 1909 was remarkably modern:

> Why have meteorologists such difficulty in predicting the weather with any certainty? Why is it that showers and even storms seem to come by chance, so that many people think it quite natural to pray for rain or fine weather, though they would consider it ridiculous to ask for an eclipse by prayer? We see that great disturbances are generally produced in regions where the atmosphere is in unstable equilibrium. The meteorologists see very well that the equilibrium is unstable, that a cyclone will be formed somewhere, but exactly where they are not in a position to say; a tenth of a degree more or less at any given point, and the cyclone will burst here and not there, and extend its ravages over districts it would otherwise have spared. If they had been aware of this tenth of a degree, they could have known it beforehand, but the observations were neither sufficiently comprehensive nor sufficiently precise, and that is the reason why it all seems due to the intervention of chance. (Poincaré, [29], pp. 68–69)

Actually, the ultimate paradox is that the existence of chaotic dynamical systems in fact supports universal determinism rather than contradicts it[11]. Suppose for a moment that no classical mechanical system can behave chaotically. That is, suppose that there

[11] Of course, since classical mechanics is not really fundamental (quantum

is a theorem saying that any such system must eventually behave in a non-chaotic fashion. It is not completely obvious what the conclusion would be, but certainly *that* would be an embarassment for the classical world-view. Indeed, so many physical systems seem to exhibit sensitivity to initial conditions that one would be tempted to conclude that classical mechanics cannot adequately describe those systems. One might suggest that there must be an inherent indeterminism in the basic laws of nature. Of course, other replies would be possible, but it is useless to speculate on this fiction since we know that chaotic behaviour is compatible with a deterministic dynamics. The only point of this story is to stress that deterministic chaos increases the explanatory power of deterministic assumptions, and therefore, according to normal scientific practice, *strengthens* those assumptions. Moreover, some deterministic chaotic system produce results that can pass all the tests used to check whether a series of numbers is "random"; this is a very strong argument against the idea that one can ever prove that some phenomenon is "intrinsically random", in the sense that there exist no deterministic mechanism underlying and explaining its behaviour. So, putting aside for a moment quantum mechanics, to be discussed in Section 5, the recent discoveries about chaos do not force us to change a single word of what Laplace wrote.

2.2. Confusions about chaos.

One often finds authors who see chaos theory as a revolution against Newtonian mechanics — the latter being labelled "linear" — or who cite quantum mechanics as an example of a nonlinear theory.[12] In actual fact, Newton's "linear thought" uses equations that are perfectly *nonlinear*; this is why many examples in chaos theory come from Newtonian mechanics, so that the study of chaos represents in fact a *renaissance* of Newtonian mechanics as a subject for cutting-edge research.

Furthermore, the relationship between linearity, chaos and an

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mechanics is), and since, as explained in Section 1, the question of 'determinism' is rather metaphysical, this issue is somewhat academic. I nevertheless want to discuss it, because there is a great deal of confusion about this in the literature.

[12]Several examples are quoted in [33], from which this section is partly taken.

equation's explicit solvability is often misunderstood. Nonlinear equations are generally more difficult to solve than linear equations, but not always: there exist very difficult linear problems and very simple nonlinear ones. For example, Newton's equations for the two-body Kepler problem (the Sun and *one* planet) are nonlinear and yet explicitly solvable. Besides, for chaos to occur, it is necessary that the equation be nonlinear and (simplifying somewhat) not explicitly solvable, but these two conditions are by no means *sufficient* — whether they occur separately or together — to produce chaos. Contrary to what people often think, a nonlinear system is not necessarily chaotic.

The difficulties and confusions multiply when one attempts to apply the mathematical theory of chaos to concrete situations in physics, biology or the social sciences. To do this in a sensible way, one must first have some idea of the relevant variables and of the type of evolution they obey. Unfortunately, it is often difficult to find a mathematical model that is sufficiently simple to be analyzable and yet adequately describes the objects being considered. These problems arise, in fact, whenever one tries to apply a mathematical theory to reality.

Some purported "applications" of chaos theory — for example, to business management or literary analysis — border on the absurd.[13] And, to make things worse, chaos theory — which is well-developed mathematically — is often confused with the still-emerging theories of complexity and self-organization.

Another major confusion is caused by mixing the mathematical theory of chaos with the popular wisdom that small causes can have large effects: "if Cleopatra's nose had been shorter", or the story of the missing nail that led to the collapse of an empire. One frequently hears claims of chaos theory being "applied" to history or society. But human societies are complicated systems involving a vast number of variables, for which one is unable (at least at present) to write down any sensible equations. To speak of chaos for these systems does not take us much further than the intuition

[13] For thoughtful critiques of applications of chaos theory in literature, see, for example, Matheson and Kirchhoff [26] and van Peer [35].

already contained in the popular wisdom.[14]

Yet another abuse arises from confusing (intentionally or not) the numerous distinct meanings of the highly evocative word "chaos": its technical meaning in the mathematical theory of nonlinear dynamics — where it is roughly (though not exactly) synonymous with "sensitive dependence on initial conditions" — and its wider senses in sociology, politics, history and even theology[15], where it is frequently taken as a synonym for disorder.

3. Probabilities in Classical Physics.

In this section, I will discuss the 'classical' notion of probability and try to dispel some of the confusions attached to that notion, confusions that are again partly due to misunderstandings related to the idea of 'determinism'.

There are, traditionally, at least two different meanings given to the word 'probability' in the natural sciences. The first notion that comes to mind is the so-called 'objective' or 'statistical' one, i.e. the view of probability as something like a 'theoretical frequency': if one says that the probability of the event E under condition X, Y, Z equals p, one means that if one reproduces the conditions X, Y, Z sufficiently often, the event E will appear with frequency p. Of

[14] Of course, if one understood these systems better — enough to be able to write down equations that describe them at least approximately — the mathematical theory of chaos might provide interesting information. But sociology and history are, at present, far from having reached this stage of development (and perhaps will always remain so).

[15] For example: "Each time that a new order of things appear, it is marked by the dissipation of a chaotic behaviour, by a broken form of movement, that is, by a "fractal", by non-linearity, by variations or "fluctuations", by instability and randomness. In this way the dynamics of self-organization of matter, which reaches the great complexity of consciousness, manifests itself"(Ganoczy, [12], p.79). The quotation appears in a section of a chapter on "God in the language of the physicists". And: ""Inventive" disorder is part of the definition of things... Impredictability which is not due to our inability to control the nature of things, but to their nature itself, whose future simply does not yet exist, and could not yet be forecasted, even by "Maxwell's demon" put on Sirius." (Gesché, [13], p. 121) This author claims to find his inspiration on the "new scientific understanding of the Cosmos" from, among others, "La nouvelle alliance" ([13], p.120). See [5] for other similar examples.

course, 'sufficiently often' is vague and this is the source of much criticism of that notion of probability. But, putting that objection aside for a moment and assuming that 'sufficiently often' can be given a precise meaning in concrete circumstances, probabilistic statements are, according to this view, factual statements that can be confirmed or refuted by observations or experiments.

By contrast, the 'subjective' or Bayesian use of the word probability refers to a form of reasoning and not (at least not directly) to a factual statement. Used in that sense, assigning a probability to an event expresses a (rational) judgment on the likelihood of that single event, based on the information available at that moment. Note that one is not interested here in what happens when one reproduces many times the 'same' event, as in the objective approach, but in the probability of a single event. This is of course very important in practice: when I wonder whether I need to take my umbrella because it will rain, or whether the stock market will crash next week, I am not mainly interested in the frequencies with which such events occur but with what will happen here and now; of course, these frequencies may be part of the information that is used in arriving at a rational judgment, but they are not typically the only information available.

How does one assign subjective probabilities to an event? In elementary textbooks, a probability is defined as the ratio between the number of favorable outcomes and the number of 'possible' ones. While the notion of favorable outcome is easy to define, the one of possible outcome is much harder. Indeed, for a Laplacean demon, nothing is uncertain and the only possible outcome is the actual one; hence, all probabilities are zeroes or ones. But *we* are not Laplacean demons[16] and it is here that ignorance enters. We try to reduce ourselves to a series of cases about which we are 'equally ignorant', i.e. the information that we do have does not allow us to favour one case over the other, and that defines the number of 'possible' outcomes. The standard examples include the throwing of a dice or of a coin, where the counting is easy, but that situation is not typical.

At the time of Laplace, this method was called the 'principle of

[16] This was of course Laplace's main point, as discussed in Section 2.

indifference'; its modern version is the maximum entropy principle. Here one assigns to each probability distribution $\vec{p} = (p_i)_{i=1}^N$ its Shannon (or Gibbs) entropy, given by:

$$S(\vec{p}) = -\sum_{i=1}^{N} p_i \ln p_i. \tag{3.1}$$

One then chooses the probability distribution that has the maximum entropy, among those that satisfy certain constraints that incorporate the information that we have about the system[17].

The rationale, like for the indifference principle, is not to introduce bias in our judgments, namely information that we do not have (like people who believe in lucky numbers). And one can reasonably argue that maximizing the Shannon entropy is indeed the best way to formalize that notion[18].

In practice, one starts by identifying a space of states in which the system under consideration can find itself and one assigns a prior distribution to it (maximizing entropy), which is then updated when new information becomes available[19].

Note that probabilistic statements, understood subjectively, are forms of reasoning, although not deductive ones[20]. Therefore, one cannot check them empirically. If someones says: Socrates is an angel; all angels are immortal; therefore Socrates is immortal, it is a valid (deductive) reasoning. Likewise, if I say that all I know about a coin is that it has two faces and that it looks symmetric, therefore the probability of 'head' is one half, it is a valid probabilistic reasoning; throwing the coin a thousand times with a result that is always tails does not disprove the reasoning; it only indicates that the initial assumption (of symmetry) was probably wrong (just as watching Socrates dead leads one to reconsider the notion that he is an angel or that the latter are immortal); the main point of

[17]In the simplest examples, the constraint is that the average of certain quantities (like the energy or the density) is fixed.

[18]See e.g. Cox [7].

[19]For an exposition of how Bayesian updating is done, see e.g. [18, 19, 20].

[20]Indeed, they are a paradigmatic example of inductive reasoning. The calculus of probabilities is deductive, of course, but the Bayesian justification of the axioms of probability is not; nor is the choice of the prior distributions.

Bayesianism is to give rules that allow to update one's probabilistic estimates, given previous observations.

Let me now discuss several objections or questions that are often raised about this 'subjective' view of probability which is more misunderstood and misrepresented than it ought to be.

1. *Commitment to subjectivism.* Some people think that a Bayesian view of probabilities presupposes of some form of subjectivism, meant as a doctrine in philosophy or philosophy of science. To make matter worse, Bayesians sometimes talk as if all of science was about 'information' and never about facts or laws. Moreover, Bayesians often stress the idea that probabilities reflect our ignorance or quantify our ignorance and that makes some physicists uneasy: putting aside parapsychology, our knowledge or our ignorance do not play a causal role in the physical world; so, why should they enter in a fundamental way in our physical theories?

   But there is no logical connection here: a subjectivist about probabilities may very well claim that there are objective facts in the world and laws governing them, and consider probabilities as being a tool used in situations where our knowledge of those facts and those laws is incomplete. In fact, one could argue that, if there is any connection between Bayesianism and philosophical subjectivism, it goes in the opposite direction; a Bayesian should naturally think that one and only one among the 'possible' states is actually realized, and that there is a difference between what really happens in the world and what we know about it. But the philosophical subjectivist position often starts by *confusing* the world and our knowledge of it (for example, much of loose talk about everything being information often ignores the fact that 'information' is ultimately information about something which itself is not information). Thus, Bayesians should not be thought of as natural fellow-travellers of philosophical subjectivists.

   Besides, ignorance does enter in the computations of probabilities but, as we will see in the next section, this does not mean that either knowledge or ignorance are assumed to play a fundamental role in physics.

2. *Commitment to determinism.* At the other extreme, subjectivists are sometimes accused of holding a deterministic view of the world, which denies the possibility of free will or of intrinsic randomness (since all probabilities are 'subjective' or 'epistemic'). But, again, this is not necessary. A Bayesian simply tends to be agnostic concerning the issue of intrinsic randomness[21] and will point out that, as I explained in the previous sections, it is difficult to find an argument showing the presence of intrinsic randomness in nature.

3. *Practical problems.* Another problem is that it is often hard to assign unambiguously a subjective probability to an event. It is easy, of course, for coin tossing or similar experiments where there are finitely many possible outcomes, which, moreover, are related by symmetry. In general, one may use maximum entropy principles, but then, one may encounter various problems: how to choose the right set of variables, how to assign an a priori distribution on those, corresponding to maximal ignorance, how to incorporate the "knowledge that we have". Many people consider such problems as objections to the subjectivist approach; but that misses the point of what this approach tries to achieve: finding the best thing to do in a bad situation, namely one where we are ignorant. Of course, the greater the ignorance, the worse the situation. To take an extreme case, we may not even realize that we are ignorant; a fortiori, there are situations where we do not know the right variables or how to put a natural prior probability distribution on them. But, so what? Bayesianism does not promise

[21]For example, Laplace wrote: "The curve described by a molecule of air or of vapour is following a rule as certainly as the orbits of the planets: the only difference between the two is due to our ignorance. Probabilility is related, in part to this ignorance, in part to our knowledge."[22] One could rephrase that statement more carefully as follows: "Even if the curve described by a molecule of air follows a rule as certainly as the orbits of the planets, our ignorance would force us to use probabilistic reasonings".

miracles; the Bayesian answer to that sort of criticism is to say: what else do you suggest to do (apart from giving up) ?

4. *(Ir)relevance to physics.* Yet another objection is: what is the physical meaning of probability one half for a single event? Bayesian thinking may be useful in bets or in various practical situations where decisions have to be made, but what role does that have in physical theories, which are supposed to describe the world as it is and not to deal with problems of practical rationality? I'll try to answer that last question in the next section.

4. The Law of Large Numbers and Scientific Explanations.

A way to make a connection between the two views on probability goes through the law of large numbers: the calculus of probabilities – viewed now as part of deductive reasoning – leads one to ascribe subjective probabilities close to one for certain events that are precisely those that the objective approach deals with, namely the frequencies with which some events occur, when we repeat many times the 'same' experiment[22].

Let me state the law of large numbers, using a terminology similar to the one used in statistical mechanics. Consider the simple example of coin flipping. Let 0 denote 'head' and 1, 'tail'. The 'space' of results of any single flip, $\{0, 1\}$, will be called the 'physical space' while the space of all possible results of N flips, $\{0, 1\}^N$, will be called the 'phase space'. The variables N_0, N_1 that count the number of heads (0) or tails (1) will be called macroscopic. Here we introduce an essential distinction between the macroscopic variables, or the macrostate, and the microstate. The microstate, for N flips, is the sequence of results for all the flips, while the macrostate simply specifies the values of N_0 and N_1. Although this example is trivial, let us draw the following analogy with statistical

[22]Of course, when we say that we repeat the 'same' experiment, or that the results of different experiments are 'independent' of each other, we also try to quantify the knowledge that we have, i.e. that we do not see any differences or any causal connections between those experiments.

mechanics: N_0 and N_1 count, for a given point in phase space, corresponding to many 'particles' (i.e. to many flips), the number of 'particles' that belong to a given subset (0 or 1) of physical space.

Now, fix a small number $\varepsilon > 0$ and call a configuration such that $|\frac{N_0}{N} - \frac{1}{2}| \leq \varepsilon$ *typical*, for that given ε, and *atypical* otherwise. Then, (a weak form of) the law of large numbers states that $\forall \varepsilon > 0$, $P(T(\varepsilon)) \to 1$ as $N \to \infty$ where $T(\varepsilon)$ is the set of typical configurations and P the product measure that assigns independent probabilities $\frac{1}{2}$ to each outcome of each flip. A more intuituive way to say the same thing is that, if we simply count the number of microstates that are typical, we find that they form a fraction of the total number of microstates close to 1, for N large.

This law allows us to make a link between probabilities and scientific explanations (the latter notion being of course hard to state precisely). A first form of scientific explanation is given by *laws*. If state A produces state B, then the occurence of B can be explained by the occurrence of A. If A is prepared in the laboratory, this kind of explanation is rather satisfactory. Of course, if B is some natural phenomena, then A itself has to be explained, and that leads to a potentially infinite regress. But, in many situations, we do not have strict laws, e.g. in coin tossing, and thus we have to see what role probabilities play in our notion of explanation. Observe first that, if we toss a coin many times and we find approximately half heads and half tails, we do not feel that there is anything special to be explained. If, however, the result deviates strongly from that average, we'll look for an explanation (e.g. by saying that the coin is biased). This leads me to make the following suggestion; first, as discussed above, probabilities enter situations where our knowledge is incomplete and Bayesian methods allow us to make the most rational predictions in those situations. Now, suppose we want to explain some phenomenon when our knowledge *of the past* is such that this phenomenon could not have been predicted with certainty. I will say that our knowledge, although partial, is *sufficient* to 'explain' that phenomenon if we would have predicted it using Bayesian computations and the information we had about the past. That notion of 'explanation' incorporates, of course, as a special case, the notion of explanation based on laws. Also, it fits with our intuition concerning the coin-tossing situation discussed above: being ignorant of any properties of the coin leads us to predict a fraction of heads or tails around one-half. Hence, such a result is not surprising or, in other words, does not “need to be explained”, while a deviation from it requires an explanation.

A basically similar form of explanation is used in macroscopic physics, for example when one wants to account for the second law of thermodynamics, the law of increase of entropy. We do not know all the microstates of, say, a gas; nor do we know their evolution. But we can assign, in a Bayesian way, a probability distribution on microstates, given some information that we have on the initial macrostate of the system. Since, for each microstate, the deterministic evolution leads to a well-defined evolution of the macrostate, we can, in principle, compute the probability, relative to our initial distribution on the microstates, of a given macrostate. If it happens that the one which is overwhelmingly probable coincides with the one which is observed, then one can say that the latter has indeed been accounted for by what we knew on the initial macrostate and by the above reasoning[23].

5. What about Quantum Mechanics?

Quantum mechanics is widely regarded as having definitively shown the failure of such ideas as ‘determinism’ or even ‘realism’[24].

[23]In fact, a similar type of explanation is used in the theory of evolution, as was noted already in 1877 by C. S. Peirce :“ Mr. Darwin proposed to apply the statistical method to biology. The same thing has been done in a widely different branch of science, the theory of gases. Though unable to say what the movements of any particular molecule of gas would be on a certain hypothesis regarding the constitution of this class of bodies, Clausius and Maxwell were yet able, eight years before the publication of Darwin’s immortal work, by the application of the doctrine of probabilities, to predict that in the long run such and such a proportion of the molecules would, under given circumstances, acquire such and such velocities; that there would take place, every second, such and such a relative number of collisions, etc.; and from these propositions were able to deduce certain properties of gases, especially in regard to their heat-relations. In like manner, Darwin, while unable to say what the operation of variation and natural selection in any individual case will be, demonstrates that in the long run they will, or would, adapt animals to their circumstances.”[28]

[24]As will be clear below, the word refers here to the idea that we can obtain

For example, Wolgang Pauli wrote:

> The simple idea of deterministic causality must, however, be abandonned and replaced by the idea of statistical causality. For some physicists... this has been a very strong argument for the existence of God and an indication of His presence in nature"[25].

I want to explain here briefly, and without too many technicalities, what might motivate such ideas, but also why no such conclusion follows.

The formalism of quantum mechanics attaches to every physical system a mathematical object, called the 'wave function', or the 'state vector', whose physical meaning is that one can use it to assign probabilities to various numbers. When we measure a certain physical quantity (called an 'observable', for example the position, the velocity, the angular momentum or the energy), for a set of systems having the same wave function, we obtain as results those numbers with frequencies approximately equal to the assigned probabilities.

One problem is that this theory, although the predicted frequencies are remarkably well confirmed experimentally, does not say anything beyond that. To see why this poses a serious problem, let us start by asking what are exactly the events that those probabilities are probabilities of. Giving different answers to that question leads to assigning different meanings to the wave function, hence different meanings to quantum mechanics. I will discuss two possible answers, corresponding to what I will call the *literal* meaning and the *implicit* one. The failure to distinguish between the two is probably the source of much confusion. The literal meaning says that the probabilities refer solely to probabilities of experiments performed in the laboratory. The problem with this interpretation is that it does not allow us to say anything about the state of the world outside the laboratory. In particular, it is an unwarranted extrapolation to claim, for example, that square of the absolute value

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an objective view of the world, at least of parts of it.

[25]Quoted by Cushing [8], where several similar remarks are mentioned and discussed.

of the wave function ($|\psi(x)|^2$) denotes the probability density of the position of the particle. It is rather, if one adheres to the literal interpretation, the probability density of *finding* the particle at x once a suitable measurement takes place. Before the measurement, it is neither here nor there.

Of course, the same argument applies to all other properties, like energy, momentum, angular momentum, spin etc.... Particles have properties like their mass or their (total) spin, but these are generic properties, not individual ones. One may of course assign a wave function to an object outside the laboratory, but the point is that the wave function does not represent the state of the system, but only the probabilities of what would happen if that object was brought into a laboratory and interacted with an apparatus there. One cannot even say that the properties of the particle are "affected" by measurements. One should rather say that they are "created" by them or that the particle and the measuring device form an "inseparable whole". Actually, if one follows the literal interpretation, talking about particles or microscopic systems should be regarded as an abuse of language. The theory speaks only about the predictable and reliable behaviour of certain macroscopic objects called measuring devices. Or, as Heisenberg wrote:

> The idea of an objective real world whose smallest parts exist objectively in the same sense as stones or trees exist, independently of whether or not we observve them ... is impossible"([17], p. 129; quoted by Goldstein in [15]).

To see how serious the problem is, consider any scientific explanation. It always ultimately follows the reductionist line: properties of "objects" are explained in terms of those of their constituents, which are ultimately governed by quantum mechanics. But here, the theory becomes suddenly silent and talks only about the behaviour of macroscopic objects, and the reductionist line becomes a circle[26].

[26] Applied to evolution, this leads to particularly strange views: evolution works through mutations involving biochemical processes that are ultimately governed by quantum mechanics. But any detailed description of these phenomena would have to refer to measurements made in present-day laboratories.

At this point, the reader may think that I exaggerate and that this is not what the standard interpretation of quantum mechanics really says. Indeed, there is a second view, which I call the implicit one, namely that "measurement" really means measurement i.e. that experiments *reveal* rather than create preexisting properties of the system[27]. According to that view, particles *have properties* such as position, momentum, angular momentum and, when we measure those quantities, we simply discover the value that the system assigned to them independently of any measurement.

If one follows that interpretation, probabilities in quantum mechanics have a conceptual status rather similar to the one in classical physics: particles have properties such as spin, position or momentum, but the latter are unknown to us; hence, we use probabilities to describe that situation. The only novel feature[28] compared to classical probabilities is that these properties might be unknowable, even in principle (or, at least, we can know only some of them at a time), because measuring one property changes the wave function (following the so-called "collapse postulate") and hence may change the values corresponding to 'observables' different from the one being measured.

The implicit view is probably what lies behind loose, but familiar, statements such as "the wave function does not represent the system but our knowledge of it" (or "quantum mechanics does not deal with Nature but with our knowledge of it"). According to that line of thought, the reduction of the wave function poses no particular problem. When we measure a system, we learn something

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As Bell emphasized: "It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? Was the wavefunction of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system... with a Ph D?" ([2], p. 117). See also [1] for a detailed critique of the centrality of the notion of "measurement".

[27]I call this view "implicit", because I believe that this is what many physicists think quantum mechanics means, although it is neither what the usual formulation implies nor, as we shall see, logically consistent.

[28]Apart from the fact that the Schrödinger equation governs the evolution of the amplitude rather than that of the probability; this is true (and often emphasized) but is not relevant for the present discussion.

about it, so our knowledge (i.e. the wave function) changes[29].

If the implicit view was tenable, there would indeed be not much to worry about. However, the problem is that it is logically inconsistent. Indeed, there are theorems[30] showing that, if one assumes that all physical quantities (or at least sufficiently many of them) have definite values before being "measured", then one runs into contradictions. In other words, what is called, in quantum mechanics, "measurement" does not do what the word suggests, namely to reveal some preexisting quantity. Are we then forced to accept the literal interpretation, and all its weird consequences? Not necessarily. The problem is with the word (measurement), not with the world! What one should look for is a theory that does complete the quantum description, i.e. that does introduce some additional parameters, beyond the wave function, but without running into the contradictions implied by the no hidden variable theorems. Moreover, such a theory should give a description of the processes usually called "measurements" that would show that they are genuine interactions, or, as is often said, that the measuring devives play an active role, instead of passively recording "what is there". But is there any hope for such a theory to exist? Could such a theory even be deterministic? The following remarks, by Max Born:

> No concealed parameters can be introduced with the help of which the indeterministic description could be transformed into a deterministic one. Hence, if a future theory should be deterministic, it cannot be a modification of the present one but must be essentially different" [4];

and by John von Neumann:

> It is therefore not, as is often assumed, a question of a reinterpretation of quantum mechanics-the present system of

[29]I believe that most physicists who do not worry about foundations of quantum mechanics often adopt this implicit view; however, a detailed sociological analysis is needed in order to evaluate that conjecture.

[30]Called the "no hiden variable theorems". See Mermin [27] for a nice discussion of those theorems and the links between them. However, Mermin's interpretation of their significance is, at least concerning locality, different from mine, which is explained in [6].

> quantum mechanics would have to be objectively false, in order that another description of the elementary processes that the statistical one be possible" [34],

are not particularly encouraging in that respect. Yet, such a theory exists, since 1952, and it is quite shoking that it is not better known and appreciated. Note however that such a theory was exactly what Einstein was hoping for, when he wrote in 1949:

> I am, in fact, firmly convinced that the essentially statistical character of contemporary quantum theory is solely to be ascribed to the fact that this (theory) operates with an incomplete description of physical systems...
>
> [In] a complete physical description, the statistical quantum theory would... take an approximately analogous position to the statistical mechanics within the framework of classical mechanics. ([11], p. 666, 672; quoted in [15]).

I will therefore give a brief sketch of this theory in the last section.

6. Bohm's theory.

In Bohm's theory[31] [3], the wave function is not the complete description of the state of the system. The latter is given by both the usual wave function and the positions of all the particles:

$$(\psi, Q_1, \ldots, Q_N),$$

for a system of N particles. It is thus postulated that positions really exist, i.e. that these are the "beables" of the theory (i.e. what exists, independently of our knowledge, by opposition to "observable"). The dynamics is also made of two pieces: one is the usual Schrödinger equation, which tells how the wave function evolves and the other is the guiding equation, which tells how the particles are guided by the wave[32]:

$$\frac{d}{dt}Q_k = \frac{\hbar}{m_k} Im \frac{\psi^* \nabla_{q_k} \psi}{\psi^* \psi}(Q_1, \ldots, Q_N) \tag{6.1}$$

[31] For an introduction to that theory, and a comparison with alternative interpretations of quantum mechanics, see [16].

[32] Which is thus often called the "pilot wave".

An important (though elementary) property of this system is *equivariance*: Assume that at some initial time $t = 0$, we have a distribution of particles with density $\rho_0 = |\psi_0|^2$. Then, at a later time, the distribution of particles obtained by solving (6.1) will be $\rho_t = |\psi_t|^2$, where ψ_t is the solution of the Schrödinger equation with initial wave function ψ_0.

Using this fact, one can show that if we assume that the distribution of the constituents of the universe satisfies, at some initial time, $\rho_0 = |\psi_0|^2$, all subsequent "measurements" will agree with the quantum mechanical predictions[33]. This is how the purely deterministic Bohm's theory explains the apparently "random" behaviour of quantum systems. One might of course worry about the nature and the justification of the statistical assumption at the initial time. This requires a long discussion (see [10]), but a brief answer is that statistical mechanics also needs some initial statistical assumptions and that the latter are not so easy to justify (actually, they are much harder to justify than the corresponding ones in Bohm's theory).

Let me now sketch how Bohm's theory clarifies the "paradoxes" arising from the no hidden variable theorems. First of all, there are no "hidden variables" in that theory other than positions. There is no value of momentum, spin, angular momentum, etc. determined prior to those specific interactions with macroscopic devices misleadingly called "measurements". One can, in Bohm's theory, analyze how those interactions take place[34] and see that the result does not depend solely on the complete state (ψ, Q) of the microscopic system but also on the way the apparatus is set up. Hence, every "measurement" of anything but position is a *genuine interaction* between the system and the apparatus. It does not merely reveal preexisting properties of the system. In other words, Bohm's theory makes concrete and mathematically precise Bohr's and other people's intuition about the impossibility to separate the system and the apparatus[35].

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[33] See [10] (and also [2], chap. 19) for a justification of this claim.

[34] See [9], [14], [2], Chap.17.

[35] A standard objection to Bohm's theory is that it is non-local. Indeed, there exists a form of action at a distance in that theory. But this objection reflects a

To conclude, I would say that not only is a refutation of determinism essentially impossible, but not the slightest argument in favour of that idea is to be found in modern physics, whether in chaos theory or in quantum mechanics. What is strange is that Bohm's theory is so widely unknown and that the mere existence of such a theory is so widely considered impossible. John Bell was one of the most lucid proponent of Bohm's theory and noboby protested better than him against that state of affairs. He exlains that, when he was a student, he had read Born's book, *Natural Philosophy of Cause and Chance* [4], where, on the basis of a misunderstanding of the significance of von Neumann's no hidden variable theorem, he was making the claim quoted above[36]. But, as Bell says, "in 1952, I saw the impossible done"; and that was Bohm's theory. He continues:

> But then why had Born not told me of this 'pilot wave'? If only to point out what was wrong with it? Why did von Neumann not consider it? More extraordinarily, why did people go on producing 'impossibility' proofs, after 1952, and as recently as 1978? When even Pauli, Rosenfeld, and Heisenberg, could produce no more devastating criticism of Bohm's version than to brand it as 'metaphysical' and 'ideological'? Why is the pilot wave picture ignored in text books? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate theoretical choice? [2], p. 160.

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misundertanding of what Bell's theorem shows: any theory, if it agrees with the quantum predictions, must be non-local so that this feature of Bohm's theory is a quality rather than a defect. Ordinary quantum mechanics avoids the charge of being non local only because it is not a "theory", at least if one interprets it as being applicable only to laboratory experiments and to remain silent about the outside world. For a more detailed discussion, see e.g. [23, 2, 6].

[36] See [15] for more examples of similar misundertandings.

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